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# Experimental bifurcation analysis for a driven nonlinear flexible pendulum using control-based continuation

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**Summary.** We present a software toolbox that allows to apply continuation methods directly to a controlled lab experiment. This toolbox enables us to systematically explore how stable and unstable steady state periodic vibrations depend on parameters. The toolbox is implemented partly in MATLAB and partly on a dSPACE realtime controller board. Its functionality is tested on a driven mechanical oscillator with a strong impact nonlinearity, controlled with electromagnetic actuators. We show how to tune a controller so that the steady state dynamics of the controlled experiment matches that of the corresponding *un*-controlled experiment.

## Introduction

Being able to observe stable as well as unstable steady state responses directly in experiments has many interesting perspectives. It allows to perform bifurcation analysis in experiments for which no good model exist - or helps to validate and improve existing models by comparing them with experimental data for both stable and unstable dynamics. Furthermore, stable branches in bifurcation diagrams might be connected via unstable branches, thus following an unstable branches might reveal otherwise overlooked dynamics, and in the case of multi-stability, conventional parameter sweeps might not detect all stable states.

The recently developed method of *control-based continuation* makes such investigations of both unstable and stable states possible. The method was first introduced in [1] and its application was further developed in [2, 3, 4]. It allows for a direct systematic exploration of the dependency of a physical system on parameters, including tracking unstable vibrations not otherwise observable in the lab. The prerequisites for this method are measurement of the modes of interest (observability) and the possibility to control the system via actuators (controllability). This makes the method immediately applicable to actively controlled machinery, such as rotors supported in active lubricated bearings [5], since all the necessary hardware for sensing, actuation and control is already present.

The focus of our work is the development of a software toolbox that implements experimental bifurcation analysis using an already existing continuation package COCO [6]. The mechanical system that is used for testing the implementation is a driven nonlinear flexible pendulum with hardening spring-stiffness and impact. The test rig design was chosen to be simple but still have sufficiently rich nonlinear dynamic behavior. Since the system shows multi-stability and has a hard impact it is a good example for the usefulness of the method, as these effects are hard to deal with, both for theoretical and experimental approaches. The type of actuators and sensors were chosen because they can be used with rotating machinery, which facilitates the intended transition into more advanced test rigs in the field of rotor dynamics, investigating bearing properties for advanced hybrid bearings.

## Methods

### Theoretical Background

Continuation packages employ a path-following algorithm to systematically trace curves of steady state dynamic responses under variation of parameters. These curves can then be collected to produce a bifurcation diagram. Typically, path-following methods implement a predictor-corrector scheme. Starting from an initially known dynamical state, the predictor makes a small step in the tangent direction of the response curve. In the correction step the predicted point is used as an initial guess for a nonlinear solver, which is applied perpendicular to the tangent direction and corrects the state back to the curve. In order to apply a continuation package a user must provide some function that implicitly defines the dynamical response of interest, which is straightforward if a model is known.

The method of control-based continuation seeks to apply the continuation technique to a suitably controlled real experiment, making it possible to directly trace out bifurcation diagrams for physical systems. The key idea is to locally stabilize the dynamical equilibrium states of the system without perturbing them. This makes it possible to trace unstable response curves, as well as preventing the system from jumping between stable states in case of multi-stability. In order to use already existing continuation packages, it is necessary to formulate a function, a so called *zero problem*, that enables the continuation algorithm to evaluate and change the controlled experiment. This function must take a control target defining the control force exerted to the experiment as an input argument. Applying the nonlinear solver to the zero problem, convergence must imply that the system is at a *un*-controlled dynamical equilibrium. This requires the control to be *non-invasive*, which means that a control force will only be exerted when the system is *not* at a dynamical equilibrium. This implies that the steady state dynamics of the controlled experiment is identical to the steady state dynamics of the uncontrolled experiment. This idea was first introduced in [7] and later applied to continuation in [1, 2]. This paper will describe how to formulate a zero problem, enabling COCO [6] to perform control-based continuation in experiments.

Particular focus will be on the implementation and tuning of the locally stabilizing non-invasive control scheme.

To make the above conditions more precise, we consider an experiment as a process that runs over time  $t$  and depends on a number of parameters  $\mu$ , and taking a single measurement can be thought of as a function evaluation  $y(\mu, t)$ . Since we are interested in periodic states we need to sample the experiment over an interval of time covering at least one period. We represent one such sample as a finite sequence of the form

$$Y(\mu, N) = \{y_0, \dots, y_{N-1}\}, \quad (1)$$

where  $N$  denotes the number of sampled points. The measurements are taken with a constant sampling interval  $h$ , that is, that the  $k$ -th measurement is  $y_k = y(t_0 + kh)$ , where  $t_0$  denotes the starting time of the measurement. Similarly, we denote a sample of a controlled experiment as

$$Z(\mu, N, u) = \{z_0, \dots, z_{N-1}\}, \quad (2)$$

where  $u$  is a control force applied to the experiment. The control scheme must be chosen to satisfies the following conditions:

1. For zero control the controlled experiment must be identical to the original experiment:  $Z(\mu, N, 0) \equiv Y(\mu, N)$ .
2. The control scheme must be *locally stabilizing*, that is, any equilibrium state  $y$  of  $Y$  must become an asymptotically stable equilibrium state of  $Z$ . In other words, if a controlled experiment  $Z$  is initialized close to a equilibrium state of  $Y$ , then the state  $z$  must converge to the state  $y$  over time.
3. The control must be *non-invasive*, that is, the control force must satisfy the inequality  $\|u\| \leq \delta\|y - z\|$ .

All these conditions can be satisfied using a PD-Controller  $G$  with appropriately chosen gains. Let us express the control signal as

$$u(t) = G(x(t)) := PD(x(t) - z(t)) \quad (3)$$

where  $x(t)$  is a predefined control target set by the continuation algorithm, and  $z(t)$  is the measurement of the controlled experiment  $Z$  taken at time  $t$ . Furthermore, we denote a discrete Fourier transform of a sample by

$$c = \mathcal{F}(Y(\mu, N)). \quad (4)$$

Note that the Fourier transform  $\mathcal{F}$  and the number of samples  $N$  must be chosen such that (4) is independent of the starting time of the measurement frame  $t_0$ . Using the above definitions we can now construct a function

$$F(c, \mu; N) := \mathcal{F}\{Z[\mu, N, G(\mathcal{F}^{-1}(c))]\} - c, \quad (5)$$

which can be passed as a zero problem to a standard continuation package. Our control scheme is non-invasive, because (under simplifying assumptions on smoothness and noise)

$$\|u\| = \|PD(x - z)\| \leq \delta\|x - z\| \leq \delta\kappa\|\mathcal{F}(x) - \mathcal{F}(z)\| = \delta\kappa\|F(c, \mu; N)\| \quad (6)$$

holds, which implies that the control force vanishes whenever the difference between control target  $x$  and measurement  $z$  is zero. In the implementation we use  $\|F(c, \mu; N)\| < \text{TOL}$  as a criteria for convergence, and typically TOL can be chosen at the order of the measurement error.

### Experimental Setup and Implementation

Figures 1a and 1b show the experimental test rig, which consists of a mechanical system, sensors, actuators and a data acquisition- and control system. The mechanical system comprises a clamped flexible pendulum (1) which, when vibrating with large enough amplitudes, impacts a mechanical stop (2) causing an increase of stiffness. This nonlinearity causes a change of natural frequency with oscillation amplitude, and the appearance of a hysteresis loop when varying the frequency of the external excitation (figure 1c). The pendulum is mounted on a platform (3), which can be moved in the horizontal plane by means of an electromagnetic shaker (4). The displacement of the platform and the displacement of the pendulum is measured using two laser displacement sensors (5). An electromagnetic actuator (6) is mounted on each side of the pendulum mass. Using an amplifier and a power supply, the strength and direction of the magnetic field can be varied using a control signal. Data acquisition and control is realized using a computer equipped with a dSPACE DS1104 board and MATLAB/Simulink.

Figure 2 shows how the communication between different parts of the software is implemented. The tasks that have to be executed in real time run on the dSPACE board, while the continuation core algorithm runs asynchronously on the computer. The real time application generates the excitation signal that is sent to the shaker and constructs the control signal based on the difference between the control target and the measured relative displacement. The control is implemented using a standard MATLAB/Simulink PD-controller block. Communication between the computer and the board

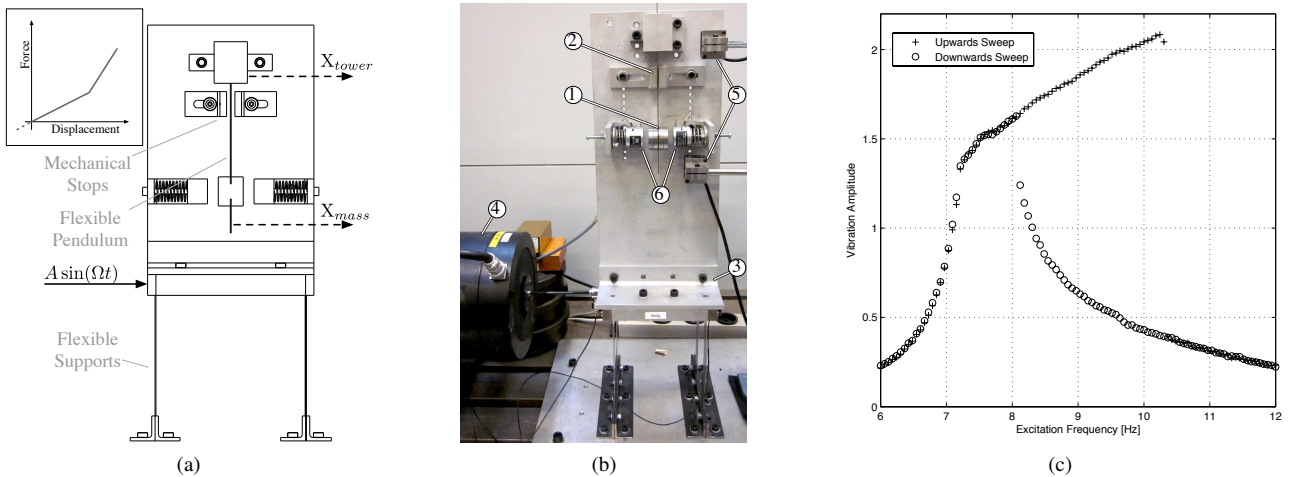


Figure 1: a) Main elements and input/output of the mechanical system. b) Experimental test rig. c) Frequency response found by frequency-sweep, keeping constant amplitude of the shaker voltage signal.

is achieved by reading and writing coefficients of Fourier modes using the MLIB/MTRACE MATLAB interface libraries provided by dSPACE. On the dSPACE card Simulink blocks are implemented in order to both compose and decompose periodic signals from and to their approximated Fourier coefficients in real time. The computer also runs dSPACE ControlDesk, which is used to monitor different parameters during the experiments.

An important task is the tuning of the gains for the PD-controller. Conventionally the tuning process is performed using a model of the physical system to be controlled. However, since the method presented intends to investigate properties of dynamical systems without models, inherently this approach cannot be used in our experiment. The gains are experimentally adjusted to constitute a control that meets a number of criteria: Firstly the control should never destabilize or disturb the equilibrium states. Secondly the control should be aggressive, meaning that it should have short reaction time and exerting large forces when the state deviates from the control target. This constitutes two competing targets and therefore the goal of the tuning process is to find a suitable compromise between control aggressiveness and non-invasiveness.

Figure 3 shows a number of parameter sweeps made on the control gains. The top row illustrates the level of invasiveness of the control. The results were obtained by measuring a stable steady state response to a certain excitation frequency and then setting this response as control target, while keeping the external excitation and varying the control gains. The plots show the resulting control signal, which for complete non-invasiveness should be zero (as is the case for zero control gains). The bottom row presents a measure for the aggressiveness of the control. A zero control target was chosen in order to keep the pendulum at the down-hanging static equilibrium while disturbing it with a harmonic excitation from the shaker and varying the control gains. The plots show the resulting control error simply being the amplitude of vibration for the pendulum. This can be thought of as the inverse of the aggressiveness. For both cases hysteresis was observed depending on the sweep direction and so, in order to have a conservative estimate, the maximum value is plotted. For both the invasiveness and aggressiveness, high and low plateaus are observed depending on the overall gain  $G$ . When choosing an appropriate set of gains one should aim at a low plateau for both conditions.

It is important that the continuation core algorithm can change the frequency of external excitation without causing the

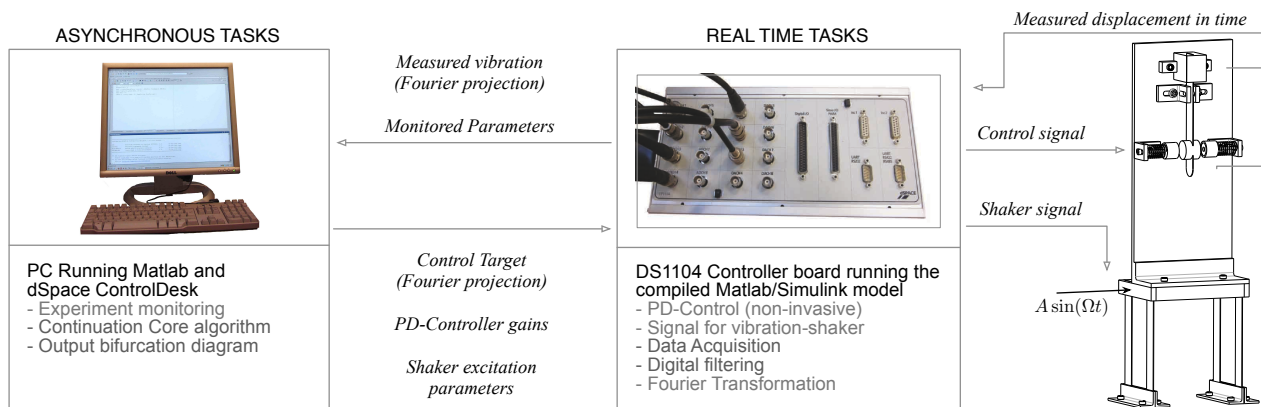


Figure 2: Overview of the communication between different parts of the hardware and software.

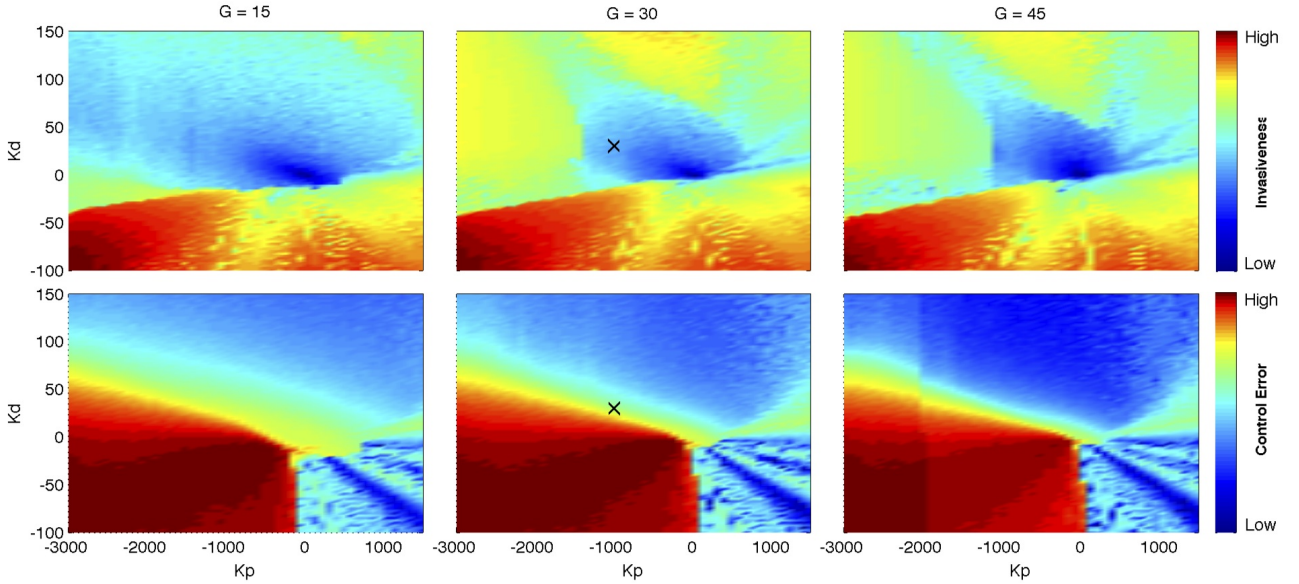


Figure 3: Experimentally obtained parameter sweeps of gains for the PD-Controller for different overall gains  $G$  applied to the controller output.  $K_p$  denotes proportional gain and  $K_d$  denotes differential gain. The top row illustrates the level of invasiveness of the control while the bottom row illustrates the aggressiveness.  $\times$  denotes the chosen gain combination for the continuation.

phase of the excitation signal to shift abruptly. Such a jump in the phase might cause unwanted perturbations of the mechanical system. Let us illustrate this: Figure 4a (top) shows an abrupt jump in phase due to two signals with very close frequencies slowly drifting in and out of phase as the experiment time elapses. The phase jump is avoided using a scaled time defined by the differential equation  $\dot{\tau} = \omega(t) \Rightarrow \tau = \int_0^t \omega(s) ds$ , which is in general not equal to  $\omega t$ . Even when the frequency is changed discontinuously, the resulting excitation will change continuously, which implies that a phase jump is avoided. The result of implementing the scaled time is shown in Figure 4a (bottom) and the implementation in Simulink is illustrated in Figure 4b.

We use a Fourier transformation applied with a shifted and averaged window function to decompose the buffered measured response into  $n$  Fourier modes. Starting from the  $k^{\text{th}}$  sample, the window  $\xi(t_k, w, t)$  of the width  $w$  holds one forcing period, that is,  $\xi(t_k, w, t) = 1$  inside and zero elsewhere. A full sampling interval  $[0, N]$  of  $N$  points can hold  $M = N - \text{floor}(1/\omega)$  such windows. The number of points in the buffer and the sampling frequency are limited by the available processing power of the dSPACE card, and thus the frequency range that is possible to measure has an upper and a lower bound. These bounds are determined by the number of points stored in the buffer and the sampling rate, because at least one whole period must fit in the buffer ( $w \leq N$ ) and high frequency oscillations must be sampled with a sufficient number of measurements in order to avoid aliasing. The Fourier coefficients are computed as

$$c_n = \int_{t_0}^{t_0+hN} \psi(t)y(t)\varphi_n(t)dt, \quad \text{where} \quad \psi(t) = \frac{1}{M} \sum_{k=0}^{M-1} \xi(t_k, k, t) \quad (7)$$

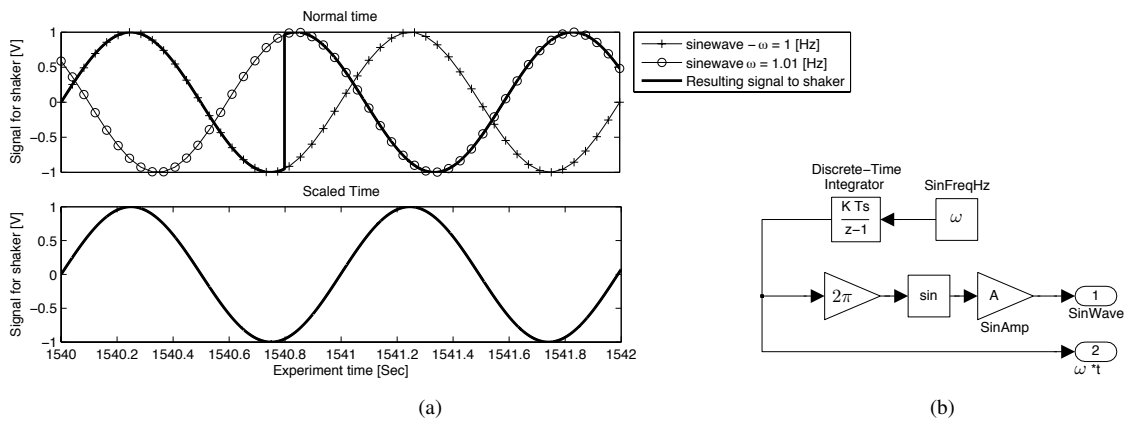


Figure 4: Example of the phase jump problem and the implemented solution. (a) Shows the effect of incrementing the frequency of excitation by 1% for the system without scaled time (top) and with scaled time (bottom). (b) Shows the Simulink block used for implementing the scaled time.

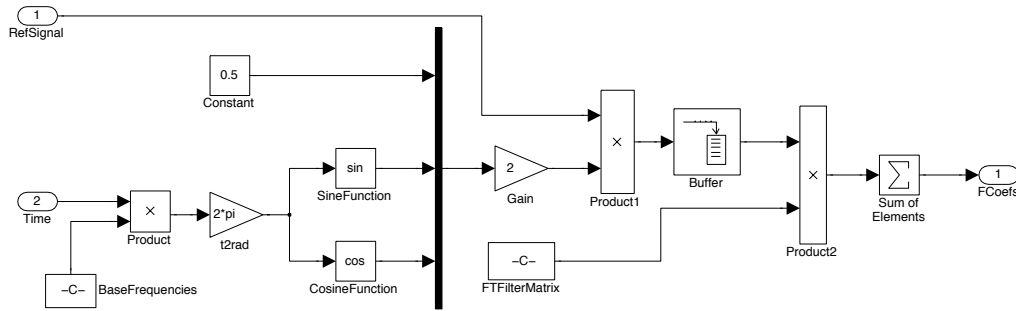


Figure 5: Simulink implementation of Fourier transform with a shifted averaged window.

where  $\varphi_n(t)$  denotes the  $n^{\text{th}}$  Fourier base functions that is projected onto and  $h$  is the sampling interval. The Fourier transform is implemented using the block shown in figure 5. Note that the averaged window can be calculated *a priori* and loaded as a matrix into the model.

## Results

Figures 6 illustrates the frequency and amplitude responses of the testrig. Figure 6a shows the result of a series of parameter sweeps for the amplitude of the signal sent to the shaker. As expected the response amplitude (calculated as the euclidian norm of the Fourier coefficients) is seen to jump abruptly at the point where the pendulum starts impacting the stops. The amplitude at which the jump occurs is seen to change with the excitation frequency. Finally, for increasing excitation frequencies, hysteresis is seen depending on the sweep direction. Figure 6 shows the result of corresponding series of frequency sweeps. A similar hysteresis behavior is observed for increasing amplitudes, the lowest amplitude essentially represents a linear system, since the amplitude of the pendulum is never sufficient to hit the stops. Note that the sweeps only find stable responses and will not necessarily find all stable responses in case of multiple stability.

Figure 7 shows the frequency response found by a continuation run plotted on top of the corresponding frequency sweep. It should be noted that there is very good agreement between the results obtained by the two methods and that the continuation is able to track the unstable part of the response curve connecting the two stable parts. At high amplitudes ( $> 1.8$ ) the influence of higher order bending modes seems to become important and quasi periodic dynamics is observed, giving rise to a number of smaller hysteresis loops along the curve. This serves as a good example for the usefulness of the method, as these phenomena would probably not have been captured studying a simple two degree of freedom piecewise linear model of the system. When the system is on a equilibrium state only small amounts of control force are exerted and we also observe that the control is normally seen not to have any low frequency content. Since the method stabilizes unstable states we have no other ways to determine the stability of a current state, than turning off the control and observe if the system diverges from the dynamical state. However, this might both cause damage to the experimental testrig and the available control energy might not be sufficient to return to the branch and resume the continuation run. This presents a problem to be tackled in our future work.

## Discussion

A suitably simple controlled experiment with sufficiently rich nonlinear dynamics was set up. A toolbox for using an existing continuation software to track stable and unstable branches of bifurcation diagrams directly in an experiment was

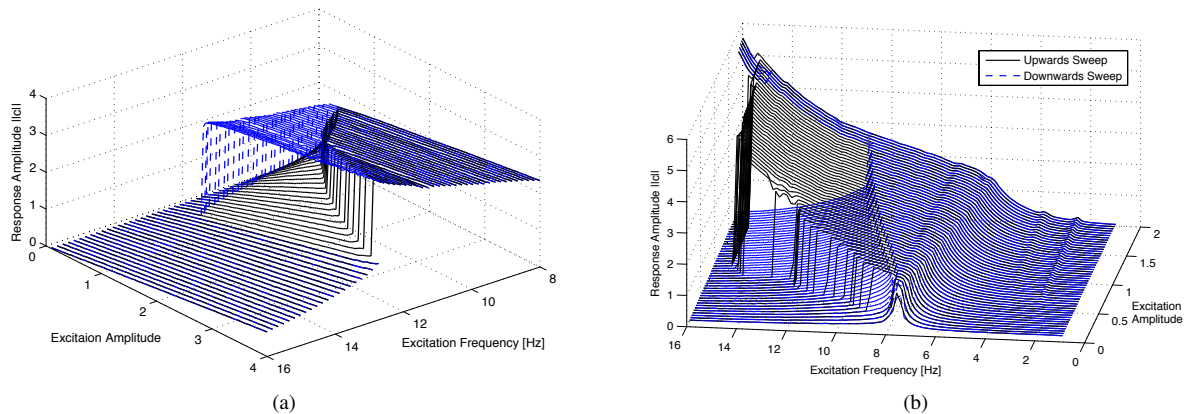


Figure 6: Experimental steady state response amplitudes recorded by parameter sweeps made by keeping one parameter fixed in steps. a) Amplitude sweep for fixed steps in frequency. b) Frequency sweep for fixed steps in amplitude.



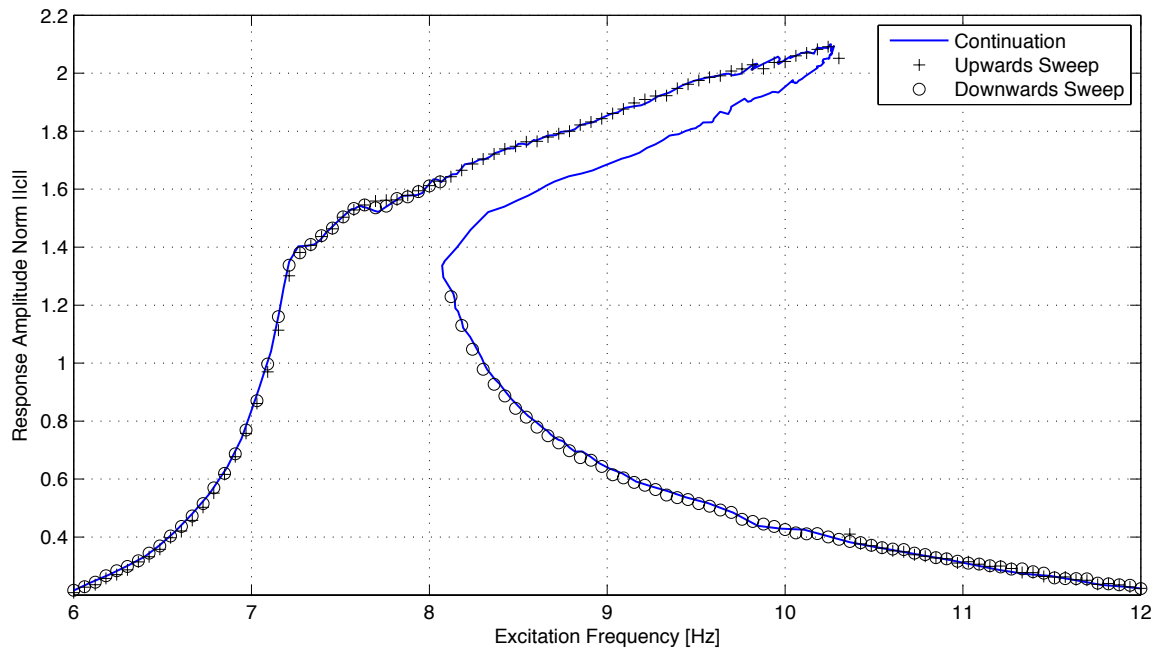


Figure 7: Bifurcation diagram (frequency response) for the driven pendulum observed experimentally by control-based continuation. The whole diagram was traced in one continuous run. The sweep was made for a excitation signal with amplitude,  $A = 0.4$ .

successfully developed and tested. In order to make the controlled experiment match the steady state dynamics of the corresponding *un*-controlled experiment, a method for tuning locally stabilizing non-invasive control was presented.

Perspectives to be tackled in future work includes; determination of stability, development of auto- or systematic tuning of control-parameters while avoiding unstable control, adaptive control during continuation runs, determination of bifurcation-points and branch switching at such point. Finally we aim at applying the method in the field of rotor dynamics, in order to explore vibrations dependency on properties of active and hybrid bearings, such as active and passive magnetic bearings, actively lubricated bearings and gas foil bearings.

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